JOURNAL OF APPROXIMATION THEORY **88**, 109–134 (1997)
ARTICLE NO. AT963015

Erdős–Turán-Type Theorems on Piecewise Smooth Curves and Arcs

Vladimir Andrievskii and Hans-Peter Blatt

Katholische Universität Eichstätt, Ostenstrasse 26–28, D-85072 Eichstätt, Germany

Communicated by V. Totik

Received May 16, 1995; accepted in revised form January 25, 1996

DEDICATED TO THE MEMORY OF J. ULLMANN

If L is a Jordan curve or a Jordan arc and p_n is a monic polynomial of degree n we obtain estimates for the discrepancy between the equilibrium measure μ_L of L and the distribution ν_{p_n} of the zeros of p_n based on one-sided bounds for the difference $U(\mu_L-\nu_{p_n}, z)$ of their logarithmic potentials. These new estimates generalize known results to the case that L is not smooth, i.e., corners of L are allowed, but cusps are not. Moreover, the results are independent of the angles at the corners. The method of proof shows that both situations—upper *or* lower bounds of $U(\mu_L-\nu_{p_n}, z)$—can be treated simultaneously. As an application, the distribution of Fekete points and extremal points of best uniform approximants can be investigated generalizing results of Kleiner [14] and Blatt and Grothmann [6] to Jordan curves and arcs with corners.

1. INTRODUCTION

Let $L \subset \mathbb{C}$ be a bounded Jordan curve or Jordan arc and let σ be a signed measure on L. The discrepancy of σ is defined by

$$D[\sigma] := \sup |\sigma(J)|,$$

where the supremum is taken over all subarcs $J \subseteq L$. In applications, frequently the signed measure σ is the difference between the equilibrium measure $\mu = \mu_L$ of L [21] and the normalized zero counting measure $\nu = \nu_{p_n}$ of a polynomial $p_n \in \Pi_n$, where Π_n denotes the set of all algebraic polynomials of degree n, i.e., the measure which associates the mass $1/n$ with each of the zeros of p_n, where each zero is counted according to its multiplicity. The discrepancy can be estimated in terms of bounds for the logarithmic potential of $\mu - \nu$,

$$U(\mu-\nu, z) := \int \log \frac{1}{|z-t|}\, d(\mu-\nu)(t).$$

0021-9045/97 $25.00

The first result in this direction was given by Erdős and Turán [9]. They considered the distribution of the zeros of a monic polynomial $p_n \in \Pi_n$, based only on its Chebyshev-norm on L,

$$\|p_n\|_L := \max_{z \in L} |p_n(z)|.$$

If L is the interval $[-1, 1]$ and

$$\|p_n\|_L \leqslant A_n/2^n = A_n(\operatorname{cap} L)^n \tag{1.1}$$

then the logarithmic potential of $\mu - \nu$ has an *upper bound*, namely

$$U(\mu - \nu, z) \leqslant \frac{\log A_n}{n} \qquad (z \in \mathbb{C}),$$

(see [6]) and the discrepancy can be estimated by

$$D[\mu - \nu] \leqslant \frac{8}{\log 3} \sqrt{\frac{\log A_n}{n}} \tag{1.2}$$

if all zeros of p_n are located on $[-1, 1]$. If L is a circle this problem was solved again by Erdős and Turán [10]. For sufficiently smooth L analogue results are due to Sjögren [18] and Blatt and Grothmann [6].

In Blatt [5] it is shown that the estimate (1.2) can be considerably sharpened if the zeros x_i are simple and this "simplicity" can be quantified in a way that

$$|p_n'(x_i)| \geqslant \frac{1}{B^n} \frac{1}{2^n} = \frac{1}{B_n} (\operatorname{cap} L)^n \tag{1.3}$$

for all zeros x_i of p_n. The basis of the proof was the reformulation of (1.1) and (1.3) as a *two-sided bound* of the logarithmic potential. For doing this, let Φ denote the conformal mapping of the unbounded component Ω of $\overline{\mathbb{C}} \backslash L$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, onto

$$\Delta := \{t \in \overline{\mathbb{C}} : |t| > 1\}$$

such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. Let $G(z)$ be the Green's function of Ω, then

$$G(z) = \log |\Phi(z)| \qquad (z \in \mathbb{C}).$$

For any $\delta \geqslant 0$, let

$$L_{1+\delta} := \{z \in \mathbb{C}: G(z) = \log(1+\delta)\}$$

denote the level curve of the Green's function G. Then the conditions (1.1) and (1.3) yield for any $k > 0$ the *two-sided bound*

$$|U(\mu - \nu, z)| \leqslant c \frac{\log C_n}{n}, \qquad C_n := \max(n, A_n, B_n),$$

for all z with $G(z) \geqslant \log(1+\delta_n)$ where $\delta_n = 1 + n^{-k}$ and c is a constant depending on k, but independent of n. If L is sufficiently smooth and $\log C_n/n$ is less than a fixed constant less than 1, then the discrepancy estimate

$$D[\mu - \nu] \leqslant c \frac{\log C_n}{n} \log \frac{n}{\log C_n}$$

holds (see Blatt [5], Totik [20], and Blatt and Mhaskar [7]).

It is therefore quite natural to ask for discrepancy estimates if, instead of an upper bound, a lower bound for the logarithmic potential $U(\mu - \nu, z)$ is known. In this context this question seems to be new although Blatt *et al.* [8] have used quantified "simplicity estimates" of zeros of polynomials to get asymptotic estimates about the distribution of extremal points in Chebyshev approximation. A basic tool in [8] was a technique of Kleiner [14] which he used to obtain results for the distribution of Fekete points on smooth Jordan curves.

This paper has now two main objects:

(1) to derive discrepancy estimates if the logarithmic potential $U(\mu - \nu, z)$ has a one-sided bound, and

(2) to generalize the discrepancy results for piecewise smooth Jordan curves and arcs.

It turns out that a unified treatment for discrepancy estimates can be obtained for both types of bounds on $U(\mu - \nu, z)$. Surprisingly, the theorems show that corners of L play no role if cusps are excluded. In [11] Grothmann has given discrepancy results depending on the angles at the corners. But in Grothmann's theorems the zeros of the polynomials are not restricted on the curve or the arc L, contrary to our assumptions. The discrepancy results of Kleiner for Fekete-points and estimates for the distribution of extremal points of best uniform approximants (see [6]) on smooth curves are special cases of our main Theorem 1. Since it is difficult to follow the original proof of Kleiner [14] we think that our proof provides easier access to his interesting results.

2. MAIN DEFINITIONS AND RESULTS

If L is a Jordan curve we denote by $\operatorname{int} L$ the bounded component and by $\operatorname{ext} L$ the unbounded component of $\overline{\mathbb{C}} \setminus L$. Let $\Psi := \Phi^{-1}$, $\operatorname{cap} L := \Psi'(\infty)$.

Following [17] the smooth Jordan curve L is called Dini-smooth if the angle $\beta(s)$ of the tangent, considered as a function of the arc length s, satisfies

$$|\beta(s_2) - \beta(s_1)| < h(s_2 - s_1) \qquad (s_1 < s_2),$$

where $h(x)$ is an increasing function for which

$$\int_0^1 \frac{h(x)}{x}\, dx < \infty. \tag{2.1}$$

In the following we will use an obvious geometrical fact:

If L is a Dini-smooth curve then for any point $z(s) \in L$, the relation

$$\begin{aligned} L \cap \{z : |z - z(s)| \leqslant \varepsilon\} \\ \subset \{z = z(s) + re^{i\theta} : 0 \leqslant r \leqslant \varepsilon,\ |\theta - \beta(s)| \leqslant h(cr) \\ \text{or } |\theta + \pi - \beta(s)| \leqslant h(cr)\} \end{aligned} \tag{2.2}$$

holds with some constants $\varepsilon > 0$ and $c > 0$ independent of s.

We call a Jordan arc Dini-smooth if it is a subarc of some Dini-smooth curve. Finally, a Jordan curve or arc is called piecewise Dini-smooth if it consists of a finite number of Dini-smooth arcs which form non-zero angles at their corners. Thus, by definition a piecewise Dini-smooth curve or arc has no cusps.

Let $p(z) = p_n(z) = \prod_{i=1}^n (z - z_i)$, be a monic polynomial of degree $n \in \mathbb{N}$. We associate with p the normalized counting measure $\nu = \nu_p$ of its zeros and assume in the following that all $z_i \in L$.

If L is a curve we consider an additional conformal mapping φ of $\operatorname{int} L$ onto the unit disk D with $\varphi(z_0) = 0$, $\varphi'(z_0) > 0$, where $z_0 \in \operatorname{int} L$ is a fixed point. Let

$$L_{1-\delta} := \{z : |\varphi(z)| = 1 - \delta\} \qquad (0 < \delta < 1),$$

then our basic results will be formulated in terms of

$$\begin{aligned} C_\delta &:= \begin{cases} L_{1+\delta} & \text{if } L \text{ is an arc} \\ L_{1+\delta} \cup L_{1-\delta} & \text{if } L \text{ is a curve,} \end{cases} \\ a_{\sup}(\delta) &:= \sup_{z \in C_\delta} U(\mu - \nu, z), \\ a_{\inf}(\delta) &:= \inf_{z \in C_\delta} U(\mu - \nu, z). \end{aligned}$$

Note that $a_{\sup}(\delta)\geqslant 0$, $a_{\inf}(\delta)\leqslant 0$ and let

$$a(\delta)=\min\{a_{\sup}(\delta),\ -a_{\inf}(\delta)\}.$$

THEOREM 1. *Let L be a piecewise Dini-smooth curve with all inner angles $\leqslant\pi$, or let L be an arbitrary piecewise Dini-smooth arc. Then there exist constants $c=c(L)>0$ and $0<\delta_0=\delta_0(L)<1/e$ such that for $0<\delta<\delta_0$,*

$$D[\mu-\nu]\leqslant c\ \sqrt{\delta\log\frac{1}{\delta}}\left(\frac{a(\delta/2)}{\delta}+1\right). \tag{2.3}$$

COROLLARY 1. *Let L be as in Theorem* 1. *Suppose that the monic polynomial $p=p_n\in\Pi_n$, $n\geqslant 2$, satisfies at least one of the following two conditions*:

(i) $\|p\|_L\leqslant A_n(\operatorname{cap} L)^n$ *with* $2\leqslant A_n\leqslant e^{n/e}$;

(ii) *all zeros $z_1, ..., z_n$ of p are simple and*

$$|p'(z_j)|\geqslant\frac{1}{A_n}(\operatorname{cap} L)^n \qquad (j=1, ..., n)$$

with $n<A_n\leqslant e^{n/e}$.

Then there exists a constant $c>0$ dependent only on L such that

$$D[\mu-\nu]\leqslant c\ \sqrt{\delta_n\log\frac{1}{\delta_n}}, \tag{2.4}$$

where

$$\delta_n:=\frac{\log A_n}{n}.$$

Indeed, reasoning in a standard way (see [5, 20, 7]) we get

$$a_{\sup}(\delta/2)\leqslant\frac{1}{n}\log A_n \qquad (\delta>0)$$

for the polynomial p satisfying condition (i), and

$$-a_{\inf}(\delta/2)\leqslant\frac{c_1}{n}\left(\log A_n+\log\frac{1}{\delta}+n\,\delta\right) \qquad (0<\delta<1/e)$$

for p satisfying (ii), where $c_1=c_1(L)>0$. Hence (2.3) with $\delta:=\delta_n$ yields (2.4).

THEOREM 2. *Let L be an arbitrary piecewise Dini-smooth curve, $z_0 \in \operatorname{int} L$ be fixed. Then there exist constants $c = c(L, z_0) > 0$ and $0 < \delta_0 = \delta_0(L, z_0) < 1/e$ such that for $0 < \delta < \delta_0$,*

$$D[\mu - \nu] \leqslant c \sqrt{\delta \log \frac{1}{\delta}} \left(\frac{2a_{\sup}(\delta/2) - U(\mu - \nu, z_0)}{\delta} + 1 \right),$$

$$D[\mu - \nu] \leqslant c \sqrt{\delta \log \frac{1}{\delta}} \left(\frac{-2a_{\inf}(\delta/2) + U(\mu - \nu, z_0)}{\delta} + 1 \right).$$

COROLLARY 2. *Let L be as in Theorem 2, and let a monic polynomial $p = p_n \in \Pi_n$, $n \geqslant 2$, satisfy one of the conditions* (i), (ii) *from Corollary* 1. *Then inequality* (2.4) *holds with*

$$\delta_n := \min \left(1/e, \frac{1}{n} \log \frac{A_n^2 (\operatorname{cap} L)^n}{|p_n(z_0)|} \right)$$

in the case (i), *and*

$$\delta_n := \min \left(1/e, \frac{1}{n} \log \frac{A_n^{c_1} |p_n(z_0)|}{(\operatorname{cap} L)^n} \right), \qquad c_1 = c_1(L, z_0) > 1,$$

in the case (ii).

First, we want to discuss the sharpness of the above results: Let $L = [-1, 1]$. According to (1.2), for any polynomial p_n satisfying (i) the inequality (2.4) is sharp up to the logarithmic term.

For polynomials p_n with property (ii) our estimate coincides with the result of Kleiner [14]. In this case it can be shown that at least the estimate $c\sqrt{\delta_n}$ for the discrepancy $D[\mu - \nu]$ cannot be improved. The question, whether the logarithmic term is necessary, is still open. Indeed, let $2 \leqslant A_n \leqslant e^{n/e}$ be arbitrary. Consider the monic polynomial

$$P_n(x) := \frac{(1 - \delta_n/2)^n}{2^{n-1}} \cos \left(n \arccos \frac{x}{1 - \delta_n/2} \right) \qquad (|x| \leqslant 1 - \delta_n/2).$$

At the zeros $x_i \in E_n := [-1 + \delta_n/2, 1 - \delta_n/2]$, $i = 1, ..., n$, of P_n we have

$$|P_n'(x_i)| = \frac{(1 - \delta_n/2)^{n-1}}{2^{n-1}} n \frac{1}{\sqrt{1 - \left(\frac{x_i}{(1 - \delta_n/2} \right)^2}} \geqslant \frac{1}{A_n} \frac{1}{2^n}.$$

Hence the polynomial P_n, considered on $[-1, 1]$ satisfies the condition (ii) from Corollary 1. On the other hand,

$$D[\mu-\nu] \geqslant \mu([1-\delta_n/2, 1]) \geqslant \sqrt{\delta_n}\,.$$

Finally we want to remark that for curves and arcs with cusps the statement of Corollary 1, and consequently Theorem 1, in general is not true: In the following we denote by $c, c_1, \ldots$ positive constants, and by $\varepsilon, \varepsilon_1, \ldots$ sufficiently small positive constants, in general different at different occurrences, but only depending on the geometry of L. We shall use the notations $a \preccurlyeq b$ for $a \leqslant cb$ and $a \asymp b$ if simultaneously $a \preccurlyeq b$ and $b \preccurlyeq a$.

Consider a function $f \in C^2([0, 1])$ satisfying for $j=0, 1, 2$ conditions

$$f^{(j)}(x) > 0 \qquad (0 < x < 1),$$

$$\lim_{x \to +0} f^{(j)}(x) = 0.$$

It is easily seen that the Jordan arc $L := \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 := \{z = x + if(x) \colon 0 \leqslant x \leqslant 1\}, \qquad \Gamma_2 := [0, 1],$$

has a cusp at the origin. According to a result of Pommerenke [16] a Fekete polynomial F_n of degree n for the arc L satisfies

$$\|F_n\|_L \preccurlyeq n^2 (\operatorname{cap} L)^n. \tag{2.6}$$

Let $z_1, \ldots, z_m, m \leqslant n$, be the zeros of F_n belonging to the segment $[0, \frac{1}{2} g(1/n^2)]$, where $g := f^{-1}$. We select points $\zeta_1, \ldots, \zeta_m \in \Gamma_1$ such that

$$|z_j - \zeta_j| = d(z_j, \Gamma_1) \qquad (j = 1, \ldots, m),$$

where

$$d(A, B) := \operatorname{dist}(A, B) := \inf_{z \in A,\, \zeta \in B} |z - \zeta| \qquad (A, B \subset \mathbb{C}).$$

Let us consider the polynomial

$$p(z) = p_n(z) := F_n(z) \prod_{j=1}^{m} \frac{z - \zeta_j}{z - z_j}.$$

It is easily verified that

$$|p(z)| \leqslant |F_n(z)| \qquad (z \in \Gamma_1). \tag{2.7}$$

Furthermore, let Φ_1 be the conformal mapping of $\overline{\mathbb{C}}\setminus\Gamma_1$ on Δ with $\Phi_1(\infty)=(\infty)$, $\Phi_1(\infty)>0$. By [19, p. 181], we have for $z\in[0, g(1/n^2)]$

$$\frac{1}{n^2}\geqslant d(z, \Gamma_1)\succcurlyeq(|\Phi_1(z)|-1)^2 .$$

Hence, the well-known Bernstein–Walsh theorem yields for any $z\in[0, g(1/n^2)]$

$$|p(z)|\preccurlyeq\|p\|_{\Gamma_1}\leqslant\|F_n\|_L . \tag{2.8}$$

By our assumption, in some neighborhood of the origin we have $f(x)\leqslant x^2$. Then for sufficiently large n we obtain

$$\frac{1}{n}\leqslant g\left(\frac{1}{n^2}\right)$$

and therefore

$$\left|\frac{p(z)}{F_n(z)}\right|\leqslant\left(1+\frac{2/n^2}{g(1/n^2)}\right)^m\leqslant\left(1+\frac{2}{n}\right)^n\leqslant e^2 \tag{2.9}$$

for all $z\in[g(1/n^2), 1]$. Comparing (2.7)–(2.9), we get

$$\|p\|_L\preccurlyeq\|F_n\|_L\preccurlyeq n^2(\operatorname{cap} L)^n.$$

Hence, if n is large enough then

$$\delta_n\preccurlyeq\frac{\log n}{n}.$$

At the same time

$$\left|(\mu-\nu)\left(\left[0,\frac{1}{2}\,g\left(\frac{1}{n^2}\right)\right]\right)\right|=\mu\left(\left[0,\frac{1}{2}\,g\left(\frac{1}{n^2}\right)\right]\right)\succcurlyeq g\left(\frac{1}{n^2}\right).$$

It is obvious that $g(x)$ can tend to zero (as $x\to 0$) arbitrarily slowly and we come to the conclusion that for arcs with cusps the discrepancy of the measure $\mu-\nu$ cannot be estimated by a universal function of A_n.

3. SOME AUXILIARY FACTS FROM GEOMETRIC FUNCTION THEORY AND QUASICONFORMAL MAPPINGS

We begin with an estimate which can be derived from [13, pp. 319–320] (see also [12, p. 6]). The definitions and properties of moduli of families of curves, harmonic measures, quasiconformal curves, and quasiconformal mappings used below can be found in [1, 2, 15].

Let G be a bounded Jordan domain, and let J be a subarc on its boundary ∂G. The harmonic measure $\omega(z, G, J)$ of J at the point $z \in G$ with respect to G satisfies

$$\omega(z, G, J) \leqslant c \exp(-\pi m(\Gamma)), \tag{3.1}$$

where $m(\Gamma)$ is the module of the family Γ of all cross cuts of G separating in G arc J from the point z.

Let Ω be the unbounded domain ($\infty \in \Omega$) of the complement $\overline{\mathbb{C}} \backslash L$ with a piecewise Dini-smooth curve $L = \partial\Omega$ as boundary. Note that L is a quasiconformal curve. Therefore, the function Φ can be extended to a quasiconformal mapping $\Phi\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the extended complex plane $\overline{\mathbb{C}}$ onto itself. Hence in the study of the metric properties of the conformal mappings Φ and $\Psi := \Phi^{-1}$ we can use the following result for quasiconformal mappings.

LEMMA 3.1 [3, Lemma 1]. *Let $w = F(\zeta)$ be a K-quasiconformal mapping of the plane on itself, with $F(\infty) = \infty$, $\zeta_j \in \mathbb{C}$, $w_j := F(\zeta_j)$, $j = 1, 2, 3$, and $|w_1 - w_2| \leqslant c_1 \, |w_1 - w_3|$. Then $|\zeta_1 - \zeta_2| \leqslant c_2 \, |\zeta_1 - \zeta_3|$ and, in addition,*

$$\left|\frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2}\right| \leqslant c_3 \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^K,$$

where $c_i = c_i(c_1, K)$, $i = 2, 3$.

Since F^{-1} is also a K-quasiconformal mapping, it follows from $|\zeta_1 - \zeta_2| \leqslant c_2 \, |\zeta_1 - \zeta_3|$ that $|w_1 - w_2| \leqslant c_4 \, |w_1 - w_3|$ and

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right| \leqslant c_5 \left|\frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2}\right|^K, \tag{3.2}$$

where $c_i = c_i(c_2, K)$, $i = 4, 5$.

An immediate consequence of Lemma 3.1 is the following statement needed in Section 7.

LEMMA 3.2. *Let $0 < \delta < 1/2$; w_1 and w_2 be such that*

$$|w_1| = |w_2| = 1, \qquad |w_1 - w_2| \geqslant \varepsilon\delta.$$

Let $\tilde{w}_i := (1 + \delta)\, w_i$, $z_i := \Psi(w_i)$, $\tilde{z}_i := \Psi(\tilde{w}_i)$, $i = 1, 2$. There exist constants $c = c(\varepsilon, L)$ and $0 < \beta = \beta(L) \leqslant 1$ such that

$$\left|\frac{z_2 - \tilde{z}_2}{z_2 - z_1}\right| \leqslant c \left|\frac{z_1 - \tilde{z}_1}{z_1 - z_2}\right|^\beta. \tag{3.3}$$

Proof. Applying Lemma 3.1 and (3.2) to $F=\Phi$, we obtain for some $K=K(L)\geqslant 1$

$$\left|\frac{z_2-\tilde{z}_2}{z_2-z_1}\right|^K \leqslant \left|\frac{\delta}{w_2-w_1}\right|,$$

$$\left|\frac{z_1-\tilde{z}_1}{z_1-z_2}\right| \geqslant \left|\frac{\delta}{w_1-w_2}\right|^K.$$

Combining these inequalities we get (3.3) with $\beta=K^{-2}$.

If G is a domain bounded by a Dini-smooth Jordan curve, then each conformal mapping φ of G onto the unit disk $D:=\{w: |w|<1\}$ has a continuously differentiable extension to $\bar{G}$ with $\varphi'(z)\neq 0$. For domains bounded by a piecewise Dini-smooth curve or piecewise Dini-smooth arc this result has some consequences and generalizations, which we formulate as Lemmas 3.3–3.5 (see [17, p. 52], for more details).

LEMMA 3.3. *Let L be a piecewise Dini-smooth curve, $z_0\in L$ be a corner with inner angle $\alpha\pi$, $0<\alpha<2$.*

Then there exists a constant $\varepsilon=\varepsilon(L, z_0)>0$ such that for all points $z\in L$, $\zeta\in\bar{\Omega}$ with $|z-\zeta|\leqslant|z-z_0|\leqslant\varepsilon$ and their images $w:=\Phi(z)$, $\tau:=\Phi(\zeta)$, $w_0:=\Phi(z_0)$ the inequalities

$$c_1\left|\frac{w-w_0}{w-\tau}\right| \leqslant \left|\frac{z-z_0}{z-\zeta}\right| \leqslant c_2\left|\frac{w-w_0}{w-\tau}\right|,$$

$$c_3\,|w-w_0|^{2-\alpha} \leqslant |z-z_0| \leqslant c_4\,|w-w_0|^{2-\alpha},$$

hold with some constants $c_j=c_j(L, z_0, \varepsilon, \alpha)>0$, $j=1, ..., 4$.

Now let L be a piecewise Dini-smooth arc, i.e. L consists of a finite number of Dini-smooth arcs $l_1, ..., l_m$. Denote by z_j and z_{j+1} the end points of l_j (the end points of L are denoted by z_1 and z_{m+1}). Set

$$\Phi(z_1)=: w_1, \qquad \Phi(z_{m+1})=: w_{m+1},$$

$$\Delta_1:=\{w: |w|>1, \arg w_1<\arg w<\arg w_{m+1}\},$$

$$\Delta_2:=\Delta\setminus\bar{\Delta}_1, \Omega_i:=\Psi(\Delta_i), \qquad i=1, 2.$$

It is easy to see that the boundaries of Δ_1 and Δ_2 are quasiconformal curves. Moreover, according to [4, Lemma 1], the boundary of Ω_1 and Ω_2 is also a quasiconformal curve. Hence, the restriction $\Phi_i: \Omega_i\to\Delta_i$ of function Φ to domain Ω_i can be extended to a quasiconformal mapping $\Phi_i: \bar{\mathbb{C}}\to\bar{\mathbb{C}}$ of the whole plane on itself, a fact that allows us to use Lemma 3.1 with $F=\Phi_i$.

LEMMA 3.4. *Let L be a piecewise Dini-smooth arc as above with corners at the points z_j, $j=2, ..., m$, and angles $\alpha_{j,i}\pi$, $0<\alpha_{j,i}<2$, with respect to Ω_i. Then there exists $\varepsilon>0$ such that for all points $z\in l_{j-1}\cup l_j$, $\zeta\in\bar{\Omega}_i$, with*

$$|z-z_{j-1}|\geqslant\varepsilon, \qquad |z-z_{j+1}|\geqslant\varepsilon,$$
$$|z-\zeta|\leqslant|z-z_j|,$$

and their images $\tau_i:=\Phi_i(\zeta)$, $w_{j,i}:=\Phi_i(z_j)$, and $w_i:=\Phi_i(z)$ the inequalities

$$c_1\left|\frac{w_i-w_{j,i}}{w_i-\tau_i}\right|\leqslant\left|\frac{z-z_j}{z-\zeta}\right|\leqslant c_2\left|\frac{w_i-w_{j,i}}{w_i-\tau_i}\right|, \tag{3.4}$$

$$c_3\,|w_i-w_{j,i}|^{\alpha_{j,i}}\leqslant|z-z_j|\leqslant c_4\,|w_i-w_{j,i}|^{\alpha_{j,i}} \tag{3.5}$$

hold with some constants $c_k=c_k(L,\varepsilon,\alpha_{j,i})>0$, $k=1, ..., 4$.

To complete the distortion properties of the conformal mapping Φ in the case of an arc L we can describe its behaviour in the neighborhood of the end points z_1 and z_{m+1} of L as follows.

LEMMA 3.5. *Let L be a piecewise Dini-smooth arc as above and let z_j, $j=1, m+1$ be its end points. Then there exists a constant $\varepsilon=\varepsilon(L)>0$ such that for any $i=1, 2$, and all points $z\in L$, $\zeta\in\bar{\Omega}_i$ with*

$$|z-\zeta|\leqslant|z-z_j|\leqslant\varepsilon$$

and their images $\tau_i:=\Phi_i(\zeta)$, $w_i:=\Phi_i(z)$, $w_{j,i}:=\Phi_i(z_j)$ the inequalities (3.4) *and* (3.5) *hold with $\alpha_{j,i}=2$.*

Moreover, we need a simple special fact concerning smoothness properties of harmonic functions.

LEMMA 3.6. *Let u be a harmonic function in Δ, continuous in $\bar{\Delta}$ and satisfying for $\theta, \theta_0\in\mathbb{R}$, $0<4\delta<\varepsilon<1$, $M>0$ the inequalities*

$$|u(e^{i\theta})-u(e^{i\theta_0})|\leqslant M(|\theta-\theta_0|+\delta) \qquad (|\theta-\theta_0|\leqslant\varepsilon),$$
$$|u(e^{i\theta})|\leqslant M.$$

Then for $0<R-1<\delta$,

$$|u(Re^{i\theta_0})-u(e^{i\theta_0}|\leqslant cM\delta\log\frac{1}{\delta}$$

with some constant $c=c(\varepsilon)>0$.

Proof. By Poisson's formula,

$$|u(Re^{i\theta_0}) - u(e^{i\theta_0})| \leqslant \frac{1}{2\pi} \int_0^{2\pi} (R^2-1) \frac{|u(e^{i\eta}) - u(e^{i\theta_0})|}{1-2R\cos(\eta-\theta_0)+R^2}\, d\eta$$

$$\leqslant \frac{5M\delta}{2\pi} \int_{\theta_0-4\delta}^{\theta_0+4\delta} \frac{R^2-1}{1-2R\cos(\eta-\theta_0)+R^2}\, d\eta$$

$$+\pi\delta M \int_{\theta_0+4\delta}^{\theta_0+\varepsilon} \frac{d\eta}{\eta-\theta_0} + \frac{\pi}{2\varepsilon^2} M\delta \leqslant cM\delta \log \frac{1}{\delta}.$$

4. PROOF OF THEOREM 1

First, let L be an arc. We restrict ourselves to the case $a(\delta/2) = a_{\sup}(\delta/2)$, i.e.

$$U(\mu - \nu, z) \leqslant a(\delta/2) \qquad (z \in \operatorname{ext} L_{1+\delta/2}). \tag{4.1}$$

The other case $a(\delta/2) = -a_{\inf}(\delta/2)$ can be handled in the same way. Consider an arbitrary subarc $J \subset L$. For our purposes it is enough to establish the inequality

$$(\nu - \mu)(J) \leqslant c \sqrt{\delta \log \frac{1}{\delta}} \left(\frac{a(\delta/2)}{\delta} + 1 \right) \tag{4.2}$$

for sufficiently small δ.

By our assumption $L = \bigcup_{j=1}^m l_j$, where each l_j is a Dini-smooth arc. Denote as before by z_j and z_{j+1} the end points of l_j, and by ζ_1 and ζ_2 the end points of J.

Let $\varepsilon_1 > 0$ be a fixed such that

$$|z_j - z_{j+1}| \geqslant 4\varepsilon_1 \qquad (j = 1, ..., m). \tag{4.3}$$

We consider first the case that one of the end points of J coincides with one of the end points of L and the other endpoint of J is "far away" from corners. Let us assume that $\zeta_1 = z_1$,

$$|\zeta_2 - z_j| \geqslant \varepsilon_1 \qquad (j = 2, ..., m).$$

The most delicate situation occurs if $\zeta_2 \in l_m$. Therefore, let us investigate first this case. Since by Lemma 3.5

$$(\nu - \mu)(J) = (\mu - \nu)(L \backslash J) \leqslant \mu(L \backslash J) \leqslant c_1 |\zeta_2 - z_{m+1}|^{1/2},$$

we may assume without loss of generality that

$$|\zeta_2 - z_{m+1}| \geqslant c_2 t^2, \tag{4.4}$$

where

$$t := \sqrt{\delta \log \frac{1}{\delta}} \tag{4.5}$$

and c_2 is a constant which is large enough for our reasoning below.

Set $R := L_{1+\delta}$, $E := \operatorname{int} R$, $R' := \{w : |w| = 1+\delta\}$. Moreover, for any point $z \in \Omega$ define the following points: $w := \Phi(z)$, $w_L := w/|w|$, $w_R = (1+\delta) w_L$, $z_L = \Psi(w_L)$, $z_R := \Psi(w_R)$. Let $w_{m+1} = e^{i\theta_{m+1}} := \Phi(z_{m+1})$, and let $\tau_1 = e^{i\eta_1}$ and $\tau_2 = e^{i\eta_2}$ be points such that $\eta_2 < \theta_{m+1} < \eta_1 < \eta_2 + 2\pi$, $\Psi(\tau_1) = \Psi(\tau_2) = \zeta_2$.

By Lemma 3.5 and (4.4) for sufficiently large constant c_2

$$\eta_1 - \theta_{m+1} \asymp \theta_{m+1} - \eta_2,$$
$$|\eta_j - \theta_{m+1}| \geqslant 3t \qquad (j = 1, 2).$$

Let $\tilde{f}[(1+\delta)\, e^{i\eta}]$ be the 2π-periodic function defined for $\tau = (1+\delta)\, e^{i\eta}$ by

$$\tilde{f}(\tau) := \begin{cases} 1 & \text{if} \quad \eta_1 - t \leqslant \eta \leqslant \eta_2 + 2\pi \\ \dfrac{1}{t}(\eta - \eta_1 + 2t) & \text{if} \quad \eta_1 - 2t \leqslant \eta < \eta_1 - t \\ 0 & \text{if} \quad \theta_{m+1} \leqslant \eta \leqslant \eta_1 - 2t \\ \tilde{f}(\tau^*) & \text{if} \quad \eta_2 \leqslant \eta < \theta_{m+1}, \end{cases}$$

where $\tau^* \in R'$ is such that $\tau^* \neq \tau$, $[\Psi(\tau^*)]_L = [\Psi(\tau)]_L$. Note that by virtue of Lemma 3.5, for points $t_1, t_2 \in R'$ with $|t_1 - t_2| \geqslant \delta$ we have

$$|\tilde{f}(t_1) - \tilde{f}(t_2)| \leqslant \frac{c_3}{t}\, |t_1 - t_2|. \tag{4.6}$$

Set $f(z) = \tilde{f}[\Phi(z)]$, $z \in R$. Let $f_+(z)$ be the harmonic function in ext R and let $f_-(z)$ be the harmonic function in int R which satisfy the boundary conditions

$$f_+(z) = f_-(z) = f(z) \qquad (z \in R).$$

Set

$$\tilde{f}_+(w) := f_+[\Psi(w)] \qquad (|w| \geqslant 1+\delta),$$
$$\tilde{f}_-(w) := f_-[\Psi(w)] \qquad (1 \leqslant |w| \leqslant 1+\delta).$$

Using Lemma 3.6 for the function $u(\tau) := \tilde{f}_+((1+\delta)\tau)$, $\tau \in \Delta$, we obtain by (4.6) the following estimate for w with $1+\delta < |w| < 1+2\delta$:

$$|\tilde{f}_+(w) - \tilde{f}(w_R)| \leqslant c_4 \frac{\delta}{t} \log \frac{1}{\delta} = c_4 t. \tag{4.7}$$

It turns out that the same fact is valid for the function $\tilde{f}_-$, namely

LEMMA 4.1. *There exists a constant $c_5 > 0$ such that*

$$|\tilde{f}_-(w) - \tilde{f}(w_R)| \leqslant c_5 t \tag{4.8}$$

for $1 \leqslant |w| < 1+\delta$.

The proof of Lemma 4.1 will be given in Section 6.

Returning to the proof of (4.2), we average the functions f_+ and f_- in some special way. Set

$$\tilde{f}(w) := \begin{cases} \tilde{f}_+(w) & \text{if} \quad |w| \geqslant 1+\delta \\ \tilde{f}_-(w) & \text{if} \quad 1 \leqslant |w| < 1+\delta. \end{cases}$$

Let $K(z)$, $z \in \mathbb{C}$, be an arbitrary averaging kernel, i.e., $K \in C^\infty(\mathbb{C})$,

$$K(z) = K(|z|) \geqslant 0 \qquad (z \in \mathbb{C}),$$

$$K(z) = 0 \qquad (|z| \geqslant 1),$$

$$\int K(z)\, dm_z = 1.$$

Consider in Δ the function

$$\tilde{g}(w) := \begin{cases} \dfrac{16}{\delta^2} \displaystyle\int \tilde{f}(\tau)\, K\left(\frac{4(\tau - w)}{\delta}\right) dm_\tau & \text{if} \quad |w| > 1 + \dfrac{1}{2}\delta \\ \tilde{f}_-(w) & 1 \leqslant |w| \leqslant 1 + \dfrac{1}{2}\delta, \end{cases}$$

If $1 \leqslant |w| < 1 + \frac{3}{4}\delta$ or $|w| > 1 + \frac{5}{4}\delta$ the value of $\tilde{g}(w)$ is an average of the harmonic function $\tilde{f}_-(w)$ or $\tilde{f}_+(w)$ respectively. Hence, by the mean-value property of harmonic functions

$$\tilde{g}(w) := \begin{cases} \tilde{f}_+(w) & \text{if} \quad |w| > 1 + \frac{5}{4}\delta \\ \tilde{f}_-(w) & \text{if} \quad 1 \leqslant |w| < 1 + \frac{3}{4}\delta, \end{cases}$$

and therefore $\Delta_w \tilde{g}(w)=0$ for such w. Note that $\tilde{g}\in C^\infty(\Delta)$ and

$$0\leqslant \tilde{g}(w)\leqslant 1. \tag{4.9}$$

Moreover, according to (4.7) and (4.8)

$$|\Delta_w \tilde{g}(w)|\leqslant c_6 \frac{t}{\delta^2} \qquad \left(1+\frac{3}{4}\delta\leqslant |w|\leqslant 1+\frac{5}{4}\delta\right). \tag{4.10}$$

Finally, the function

$$g(z):=\begin{cases} \tilde{g}[\Phi(z)], & z\in\Omega \\ f_-(z), & z\in L \end{cases}$$

has in $\mathbb{C}$ partial derivatives of all orders and by Green's formula

$$\int \Delta_z g(z)\, dm_z=0. \tag{4.11}$$

Applying the technique of [6] we can establish the inequality

$$\left|\int g(d\nu-d\mu)\right|\leqslant c_7 \frac{at}{\delta}, \tag{4.12}$$

where $a:=a(\delta/2)$. In fact, by Green's formula g can be represented as

$$g(z)=g(\infty)+\frac{1}{2\pi}\int \Delta_\zeta g(\zeta)\log|z-\zeta|\, dm_\zeta \qquad (z\in\mathbb{C}).$$

We obtain according to (4.1), (4.10), and (4.11)

$$\begin{aligned}\left|\int g(d\nu-d\mu)\right| &=\frac{1}{2\pi}\left|\int (a+U(\nu-\mu,\zeta))\,\Delta_\zeta g(\zeta)\, dm_\zeta\right| \\ &\leqslant \frac{1}{2\pi}\int_\Delta (a+\widetilde{U}(\nu-\mu,w))\,|\Delta_w \tilde{g}(w)|\, dm_w \\ &\leqslant c_8 \frac{t}{\delta^2}\int_{1+(3/4)\delta}^{1+(5/4)\delta} r\int_0^{2\pi} ((a+\widetilde{U}(\nu-\mu,re^{i\theta}))\, d\theta\, dr \\ &\leqslant c_9\frac{at}{\delta},\end{aligned}$$

where $\widetilde{U}(\nu-\mu,w):=U(\nu-\mu,\Psi(w))$.

Conditions (4.8), (4.9), and (4.12) imply

$$(\nu-\mu)(J)\leqslant\int g(d\nu-d\mu)+\mu(\Gamma_{2t})+\int_{L\setminus(J\cup\Gamma_{2t})} g\,d\mu$$

$$+\int_J (1-g)\,d\nu\leqslant c_{10}\left(\frac{at}{\delta}+t\right),$$

where

$$\Gamma_r:=\Psi(\{e^{i\eta}:\eta_1-r\leqslant\eta\leqslant\eta_1\})\qquad(0<r\leqslant 3t).$$

Now we assume that both points ζ_1 and ζ_2 are in a sufficiently small neighbourhood of some of the corners, and precisely let

$$|\zeta_j-z_2|\leqslant\varepsilon_1\qquad(j=1,2).$$

In this case our reasoning will be the same as before. However, the construction of the auxiliary function $\tilde{f}$ needs some modifications.

Let L have at the point z_2 an exterior angle $\alpha\pi$ with respect to the region Ω_1, $0<\alpha\leqslant 1$, and let $e^{i\eta_j}=\tau_j:=\Phi_1(\zeta_j)$, $e^{i\theta_2}=w_2:=\Phi_1(z_2)$. We fix two points $\xi_1\in l_1$ and $\xi_2\in l_2$ such that

$$|\xi_j-z_2|\geqslant 3\varepsilon_1,\ |\xi_1-z_1|\geqslant\varepsilon_1,\ |\xi_2-z_3|\geqslant\varepsilon_1, \tag{4.13}$$

where ε_1 is a constant from (4.3) and set

$$\Phi_2(\xi_j)=:t_j=e^{i\beta_j}\qquad(j=1,2).$$

The points $(1+\delta)t_j$ divide R' into two subarcs. Denote by S' the subarc which includes the point $(1+\delta)\,w_2$. Without loss of generality we assume that

$$\beta_2<\eta_2<\eta_1<\beta_1<\beta_2+2\pi,\qquad\beta_2<\theta_2<\beta_1.$$

We choose a constant c, $1\leqslant c\leqslant 10$, such that

$$\theta_2\notin[\eta_2-4ct,\eta_2-ct]\cup[\eta_1+ct,\eta_1+4ct], \tag{4.14}$$

where as before

$$t=\sqrt{\delta\log\frac{1}{\delta}}.$$

Consider the function $\tilde{f}(\tau)=\tilde{f}((1+\delta)\,e^{i\eta})$, $\beta_2\leqslant\eta\leqslant\beta_1$, given on S' by

$$\tilde{f}((1+\delta)e^{i\eta}) := \begin{cases} 1 & \text{if} \quad \eta_2-2ct\leqslant\eta\leqslant\eta_1+2ct \\ 0 & \text{if} \quad \beta_2\leqslant\eta\leqslant\eta_2-3ct \text{ or } \eta_1+3ct\leqslant\eta\leqslant\beta_1 \\ \dfrac{1}{ct}(\eta-\eta_2+3ct) & \text{if} \quad \eta_2-3ct\leqslant\eta\leqslant\eta_2-2ct \\ \dfrac{1}{ct}(-\eta+\eta_1+3ct) & \text{if} \quad \eta_1+2ct\leqslant\eta\leqslant\eta_1+3ct. \end{cases}$$

Let f be the function, harmonic in $\overline{\mathbb{C}}\backslash S$ where $S:=\Psi(S')$, with boundary values

$$f(z):=\tilde{f}(\Phi(z)) \qquad (z\in S).$$

For $w\in\bar{\Delta}$ set $\tilde{f}(w):=f(\Psi(w))$. Then the following estimate holds.

LEMMA 4.2. *Let* $w\in S'_\delta:=\{w\in\bar{\Delta}\colon \operatorname{dist}(w,S')\leqslant\delta\}$.

(i) *If* $w_R\in S'$, *then*

$$|\tilde{f}(w)-\tilde{f}(w_R)|\leqslant c_1 t. \tag{4.15}$$

(ii) *If* $w_R\in S'_\delta\backslash S'$ *then*

$$\tilde{f}(w)\leqslant c_2 t. \tag{4.16}$$

The proof of Lemma 4.2 is given in Section 7. Using this lemma the proof of (4.2) is complete by repeating step by step our reasoning from the previous case with the new function $\tilde{f}$. But then (4.2) is proved since for an arbitrary subarc J of an arc L

$$(\nu-\mu)(J)=\sum_{j=1}^{4}(\nu-\mu)(J_j),$$

where each of J_j satisfies one of the two restrictions mentioned above.

If L is a curve, the proof of (2.3) is even simpler than in the previous case of an arc, because we don't need to take care of end points. A short sketch of the proof follows: As before we assume that (4.1) holds. Obviously, it is enough to prove (4.2) for an arbitrary subarc $J\subset L$ with end points ζ_1,ζ_2 such that

$$J\subset l_j\cup l_{j+1},\ |\zeta_k-z_i|\geqslant\varepsilon \qquad (k=1,2;\ i=j,\ j+2),$$

where $\varepsilon > 0$ is fixed and l_j, $j=1, ..., m$, are the Dini-smooth arcs with end points z_j, z_{j+1} of which the curve L consists. We may restrict ourselves to $j=1$. As above let

$$e^{i\eta_j} = \tau_j := \Phi(\zeta_j),\ j=1, 2;\ e^{i\theta_2} = w_2 := \Phi(z_2)$$

and consider the 2π-periodic function

$$\tilde{f}((1+\delta)e^{i\eta}) = \begin{cases} 1 & \text{if} \quad \eta_2 - 2ct \leqslant \eta \leqslant \eta_1 + 2ct \\ 0 & \text{if} \quad \eta_1 + 3ct \leqslant \eta \leqslant \eta_2 - 3ct + 2\pi \\ \dfrac{1}{ct}(\eta - \eta_2 + 3ct) & \text{if} \quad \eta_2 - 3ct \leqslant \eta \leqslant \eta_2 - 2ct \\ \dfrac{1}{ct}(-\eta + \eta_1 + 3ct) & \text{if} \quad \eta_1 + 2ct \leqslant \eta \leqslant \eta_1 + 3ct. \end{cases} \tag{4.17}$$

where again the constant $1 \leqslant c \leqslant 10$ satisfies (4.14) and t is given by formula (4.5). Then the same arguments of the first part of the arc case lead to the estimate (4.2).

5. PROOF OF THEOREM 2

We restrict ourselves to prove (2.5) and it is sufficient to obtain the estimate

$$(\nu - \mu)(J) \leqslant c\sqrt{\delta \log \frac{1}{\delta}} \left(\frac{2a_{\sup}(\delta/2) - U(\mu - \nu, z_0)}{\delta} + 1 \right)$$

for a subarc $J \subset L$. Moreover, we may assume without loss of generality that

$$J \subset l_1 \cup l_2, \quad |\zeta_k - z_j| \geqslant \varepsilon_1 \qquad (k=1, 2, \quad j=1, 3),$$

where ζ_1 and ζ_2 are the end points of J, $\varepsilon_1 > 0$ is fixed and satisfies (4.3). Let L have at the point z_2 an interior angle $\alpha\pi$ with respect to the domain $G := \operatorname{int} L$, $0 < \alpha < 2$.

The proof, i.e., the construction of the auxiliary functions $f_\pm$, depends essentially on the value of α. More precisely we have to distinguish two cases.

(a) $\alpha \leqslant 1$. Consider fixed points $\xi_1 \in l_1$ and $\xi_2 \in l_2$ satisfying (4.13). Let $\varphi(\xi_j) =: \tau_j$. The points $(1-\delta)\tau_1$ and $(1-\delta)\tau_2$ divide the circle $\{\tau : |\tau| = 1 - \delta\}$ into two parts. Denote by s' that one which does not include the point $(1-\delta)\,\varphi(z_2)$. Let $s := \psi(s')$ where $\psi := \varphi^{-1}$. We consider the function $\tilde{f}$ given by the formula (4.17). Set $f(z) = \tilde{f}[\Phi(z)]$, $z \in L_{1+\delta} =: R$. As above let f_+ be the harmonic extension of f to ext R. By f_- we denote the harmonic extension of the function f on R and 0 on s to $(\operatorname{int} R)\backslash s$.

Thus the construction of the function f_- is essentially different from that in the proof of Theorem 1 for curves. Nevertheless, the analogue of Lemma 4.2 can be proved too.

Now consider the functions

$$\tilde{f}(w) := \begin{cases} f_+[\Psi(w)], & |w| \geqslant 1+\delta, \\ f_-[\Psi(w)], & 1 < |w| < 1+\delta, \\ f_-[\psi(w)], & 0 \leqslant w < 1, \end{cases}$$

$$\tilde{g}(w) := \begin{cases} \dfrac{16}{\delta^2} \displaystyle\int \tilde{f}(\tau)\, K\left(\frac{4(\tau-w)}{\delta}\right) dm_\tau & \text{if } \; 1+\dfrac{3}{4}\delta \leqslant |w| \leqslant 1+\dfrac{5}{4}\delta \\ \dfrac{16}{\delta^2} \displaystyle\int \tilde{f}(\tau)\, K\left(\frac{4(\tau-w)}{\delta}\right) dm_\tau & \text{if } \; 1-\dfrac{5}{4}\delta \leqslant |w| \leqslant 1-\dfrac{3}{4}\delta \\ \tilde{f}(w) & \text{otherwise,} \end{cases}$$

$$g(z) := \begin{cases} \tilde{g}(\Phi(z)), & z \in \Omega, \\ \tilde{g}(\varphi(z)), & z \in G, \\ f_-(z), & z \in L. \end{cases}$$

To obtain the upper estimate for the quantity $(\nu-\mu)(J)$ we have to repeat the reasoning of the proof of Theorem 1 with the new function g. Following analogue arguments as used in [6] the quantity $U(\mu-\nu, z_0)$ plays a significant role and leads to the estimate in Theorem 2.

(b) $\alpha > 1$. We repeat our reasoning of the case (a), but now interchanging the roles of the domains G and Ω, the functions φ and Φ, and so forth.

6. PROOF OF LEMMA 4.1

We begin with an auxiliary fact concerning the harmonic measure with respect to the region $E := \operatorname{int} L_{1+\delta}$.

Let $T \subset L$ be an arc with end points ξ_1 and ξ_2 such that

$$\xi_1 = z_1, \qquad \xi_2 \in l_m, \qquad |\xi_2 - z_m| \geqslant \varepsilon_1.$$

Set for $\delta > 0$, $R := L_{1+\delta}$,

$$T_\delta := \{\zeta : \zeta \in R, \zeta_L \in T\}.$$

Consider the function

$$\omega(z) := \omega(z, E, T_\delta) \qquad (z \in E),$$

i.e., the harmonic measure of T_δ at the point z with respect to the region E.

We assert that for $z \in E \backslash L$ with $z_L \in L \backslash T$ the following inequality

$$\omega(z) \preccurlyeq \exp\left(-c\frac{b}{\delta}\right) \tag{6.1}$$

holds, where $b := \operatorname{dist}(\Phi(z), T')$,

$$T' := \{w : |w| = 1, \Psi(w) \in T\}.$$

Indeed, without loss of generality we assume that $b \geqslant 2\delta$.

Denote by $Q \subset E$ a quadrilateral whose boundary consists of

$$\gamma_1 := \{\zeta \in E : \zeta_L = \xi_2\},$$
$$\gamma_2 := \{\zeta \in E : \zeta_L = z_L\},$$

and two subarcs of the curve R.

Let $m(Q)$ be the module of Q, i.e., the module of the family of all cross cuts of Q separating in Q the sides γ_1 and γ_2.

According to (3.1) and the comparison principle,

$$\omega(z) \preccurlyeq \exp\{-\pi m(Q)\}. \tag{6.2}$$

Further, we can apply Rengel's inequality (see [15, p. 22])

$$m(Q) \geqslant \frac{s(Q)^2}{A(Q)}, \tag{6.3}$$

where $A(Q)$ is the Euclidean area of Q; $s(Q)$ is the distance between γ_1 and γ_2 in Q, i.e.,

$$s(Q) := \inf_{\gamma \in \Gamma_Q(\gamma_1, \gamma_2)} |\gamma|,$$

where $\Gamma_Q(\gamma_1, \gamma_2)$ denotes the set of all cross cuts of Q which join γ_1 and γ_2, and $|\gamma|$ denotes the length of γ. Since by Lemma 3.1

$$s(Q) \succcurlyeq |z_L - \xi_2|,$$
$$A(Q) \preccurlyeq |z_L - \xi_2|\, |\xi_2 - (\xi_2)_R|,$$

we see that, by virtue of Lemma 3.4 and (6.3),

$$m(Q) \succcurlyeq \left| \frac{z_L - \xi_2}{(\xi_2)_R - \xi_2} \right| \asymp \frac{b}{\delta}. \tag{6.4}$$

Combining (6.2) and (6.4) we get (6.1).

Writing the inequality (6.1) for the arc $L \backslash T$ instead of T we obtain the following estimate for the case $z \in E \backslash L$, $z_L \in T$,

$$1 - \omega(z) \preccurlyeq \exp\left(-c\,\frac{b'}{\delta}\right), \tag{6.5}$$

where $b' := \operatorname{dist}(\Phi(z), \partial D \backslash T')$.

Returning to the proof of Lemma 4.1, we consider three particular positions of the point $z \in E$. If $z_L \in L \backslash (J \cup \Gamma_{3t})$ then, using inequality (6.1) with $T = J \cup \Gamma_{2t}$, we obtain

$$|f_-(z) - f(z_R)| = f_-(z) \leqslant \omega(z, E, T) \preccurlyeq \exp\left(-c_1\,\frac{t}{\delta}\right) \leqslant t.$$

Similarly, if $z_L \in J$, then by (6.5) for $T = J \cup \Gamma_t$ we have

$$|f_-(z) - f(z_R)| = 1 - f_-(z) \leqslant 1 - \omega(z, E, T) \preccurlyeq t.$$

The most complicated case is $z_L \in \Gamma_{3t}$. Then we consider the set

$$M := \left\{ \zeta \in R \colon |\zeta - z_L| \leqslant c_2\,|z_L - z_{m+1}|^{1/2}\,\delta \log \frac{1}{\delta} \right\},$$

where the constant c_2 can be chosen so large that (6.1) and (6.5) hold and

$$\omega(z, E, R \backslash M) \leqslant t.$$

Finally, by (4.6) and Lemma 3.5 we obtain for $\zeta \in M$

$$|f(\zeta) - f(z_R)| \preccurlyeq \frac{1}{t}\,\frac{|\zeta - z_L|}{|z_{m+1} - z_L|^{1/2}} \preccurlyeq \frac{\delta}{t} \log \frac{1}{\delta}.$$

Therefore, the maximum principle for harmonic function yields

$$\begin{aligned} |f_-(z) - f(z_R)| &\leqslant \sup_{\zeta \in M} |f(\zeta) - f(z_R)| + \omega(z, E, R \backslash M) \\ &\preccurlyeq \frac{\delta}{t} \log \frac{1}{\delta} + t = 2t. \end{aligned}$$

This completes the proof.

7. PROOF OF LEMMA 4.2

According to Beurling's theorem (see [2, p. 43]) for all $z \in V := \overline{\mathbb{C}} \setminus S$ with properties

$$|z - z_2| \geqslant 2\varepsilon_1, \qquad \operatorname{dist}(z, S) \leqslant \varepsilon_1,$$

where ε_1 is the constant of (4.3), we have

$$f(z) \leqslant c_1 \sqrt{\operatorname{dist}(z, S)}. \tag{7.1}$$

Therefore, the inequalities (4.15) and (4.16) in the case that w satisfies

$$1 < |w| \leqslant 1 + \delta, \; |z - z_2| \geqslant 2\varepsilon_1,$$

where $z := \Psi(w)$, are simple consequences of (7.1) and the maximum principle for harmonic functions.

Furthermore, applying (7.1) or Lemma 3.6 to the function

$$u(e^{i\eta}) := f(\Psi((1+\delta)\, e^{i\eta})) \qquad (0 \leqslant \eta < 2\pi)$$

we obtain the inequalities (4.15) and (4.16) for w with $1 + \delta < |w| \leqslant 1 + 2\delta$.

Thus we have only one nontrivial case, namely

$$|z - z_2| \leqslant 2\varepsilon_1, \qquad z \in \Omega_1 \cap \operatorname{int} R.$$

We begin with the estimation of the quantity

$$|\tilde{f}(w) - \tilde{f}(w_R)| = |f(z) - f(z_R)|$$

by some expressions using the notions of harmonic measure and some special cross cuts of the domain V.

It follows from Lemma 3.1 that

$$\operatorname{diam}(\{\zeta \in (\operatorname{int} R) \cap \Omega_1 : \zeta_L = z_L\}) =: d \asymp |z_L - z_R|. \tag{7.2}$$

Without loss of generality we assume that δ and consequently d are sufficiently small. For $d < r < \varepsilon := \varepsilon_1/2$ we denote by $\gamma(r) = \gamma_z(r)$ the arc of the intersection $\{\zeta : |\zeta - z_L| = r\} \cap V$ that separates the point z from ∞. Define by $l(r) = l_z(r)$ the subarc of S which has the same end points as $\gamma(r)$. It is obvious that

$$\operatorname{diam} l(r) \preccurlyeq r.$$

Consider the Jordan domain $W \subset V$ that includes the point z and is bounded by $\partial W = l(\varepsilon) \cup \gamma(\varepsilon)$. Define for $r > 0$

$$u(r) := \sup_{\xi \in \partial W \, |\xi - z_R| \leqslant r} |f(\xi) - f(z_R)|.$$

Let $v(r) := \omega(z, W, \partial W \backslash l(r))$, $d < r < \varepsilon$, be the harmonic measure of the arc $\partial W \backslash l(r)$ at the point z with respect to W. For convenience let $v(d) := 1$. Hence, choosing a natural number k such that

$$\frac{\varepsilon}{2} \leqslant 2^k d < \varepsilon$$

we get by the maximum principle for harmonic functions

$$\begin{aligned} |f(z) - f(z_R)| &\leqslant u(\operatorname{diam} l(d)) \\ &\quad + \sum_{j=0}^{k-1} v(2^j d)\, u(\operatorname{diam} l(2^{j+1} d)) + v(\varepsilon/2) \\ &\leqslant u(cd) + 2 \int_d^{\varepsilon} \frac{u(cr)}{r}\, v\left(\frac{r}{2}\right) dr + v(\varepsilon/2). \end{aligned}$$

Denote by $\Gamma(r)$, $d < r < \varepsilon$, the family of all cross cuts of W which separate the point z from $\partial W \backslash l(r)$. By (3.1) we obtain

$$v(r) \preccurlyeq \exp(-\pi m(\Gamma(r))), \tag{7.3}$$

where $m(\Gamma(r))$ is the module of the family $\Gamma(r)$. By the integrated version of the composition laws (see [2, p. 56])

$$m(\Gamma(r)) \geqslant \int_d^r \frac{dx}{|\gamma(x)|},$$

where $|\gamma(x)|$ denotes the length of $\gamma(x)$. Together with (7.3) this implies

$$\begin{aligned} &|f(z) - f(z_R)| \\ &\quad \preccurlyeq u(2cd) + \int_{2d}^{\varepsilon} \frac{u(cr)}{r} \exp\left(-\pi \int_d^{r/2} \frac{dx}{|\gamma(x)|}\right) dr + \exp\left(-\pi \int_d^{\varepsilon/2} \frac{dx}{|\gamma(x)|}\right). \end{aligned}$$

Two immediate conclusions can be made. Namely, it is easy to see that for any z

$$u(2cd) \preccurlyeq \frac{\delta}{t} \leqslant t.$$

Further, since

$$|\gamma(x)| \leqslant 2\pi x,$$
$$d \leqslant \delta,$$

we obtain

$$\exp\left(-\pi \int_d^{\varepsilon/2} \frac{dx}{|\gamma(x)|}\right) \leqslant \left(\frac{2d}{\varepsilon}\right)^{1/2} \leqslant \delta^{1/2} \leqslant t.$$

By (2.2), Lemma 3.1, and Lemma 3.2 we have for $d < x < \varepsilon$ and some $0 < \beta < 1$

$$\varepsilon_2 x \leqslant |\gamma(x)| \leqslant \pi(x + c_1 x h(c_2 x) + c_3 d^\beta x^{1-\beta}),$$

where h is the function from the definition of Dini-smooth curve. Hence by (2.1)

$$\begin{aligned}\pi \int_d^{r/2} \frac{dx}{|\gamma(x)|} &= \int_d^{r/2} \frac{dx}{x} + \int_d^{r/2} \frac{\pi x - |\gamma(x)|}{x|\gamma(x)|}\,dx \\ &\geqslant \log \frac{r}{2d} - \frac{c_1}{\varepsilon_2} \int_0^{r/2} \frac{h(c_2 x)}{x}\,dx - \frac{c_3}{\varepsilon_2} d^\beta \int_d^{r/2} \frac{x^{1-\beta}}{x^2}\,dx \\ &\geqslant \log \frac{r}{d} - c_4,\end{aligned}$$

and, consequently,

$$\begin{aligned}\int_{2d}^{\varepsilon} \frac{u(cr)}{r} \exp\left(-\pi \int_d^{r/2} \frac{dx}{|\gamma(x)|}\right) dr &\leqslant d \int_{2d}^{\varepsilon} \frac{u(cr)}{r^2}\,dr \\ &= cd \int_{2cd}^{\varepsilon c} \frac{u(r)}{r^2}\,dr.\end{aligned}$$

In order to establish inequality (4.15) we have therefore to show that

$$d \int_d^{\varepsilon_3} \frac{u(r)}{r^2}\,dr \leqslant t. \tag{7.4}$$

Let $w_2 := \Phi_1(z_2)$ and consider two particular positions of the point z.

First assume that $|w_L - w_2| \geqslant t/2$ holds. By our assumption and Lemma 3.3, $d < |z_L - z_2|$ and for $d < r < |z_L - z_2|$,

$$u(r) \leqslant \frac{r}{t} \frac{|w_L - w_2|}{|z_L - z_2|}.$$

Thus we have owing to Lemma 3.1

$$d\int_d^{\varepsilon_3}\frac{u(r)}{r^2}\preccurlyeq d\int_d^{|z_L-z_2|}\frac{u(r)}{r^2}\,dr+d\int_{|z_L-z_2|}^{\varepsilon_3}\frac{dr}{r^2}$$

$$\preccurlyeq\frac{d}{|z_L-z_2|}\,|w_L-w_2|\,\frac{1}{t}\log\frac{|z_L-z_2|}{d}+\frac{d}{|z_L-z_2|}. \tag{7.5}$$

Note that by (7.2) and Lemma 3.3

$$\frac{d}{|z_L-z_2|}\asymp\left|\frac{z_L-z_R}{z_L-z_2}\right|\asymp\frac{\delta}{|w_L-w_2|}. \tag{7.6}$$

Combining (7.5) and (7.6) we get (7.4).

Now let $|w_L-w_2|<t/2$. Then again by Lemma 3.3 and the construction of function f we obtain

$$u(r)=0 \qquad \text{for} \quad 0<r<\varepsilon_4 t^{2-\alpha}.$$

Therefore, as in the previous case, we have

$$d\int_d^{\varepsilon_3}\frac{u(r)}{r^2}\,dr\leqslant d\int_{\varepsilon_4 t^{2-\alpha}}^{\varepsilon_3}\frac{dr}{r^2}\leqslant\frac{1}{\varepsilon_4}\,\frac{d}{t^{2-\alpha}}\preccurlyeq\frac{\delta}{t}\leqslant t,$$

where we have used the inequality

$$\frac{d}{t^{2-\alpha}}\preccurlyeq\frac{\delta}{t}. \tag{7.7}$$

In order to prove (7.7) we consider the points $w_L=e^{i\theta_L}$, $w_2=e^{i\theta_2}$ and assume for definiteness that

$$\theta_2<\varepsilon_L<\theta_2+t.$$

Setting $\tau:=e^{i(\theta_2+2t)}$, $\zeta:=\Psi(\tau)$, it follows by Lemma 3.1 and Lemma 3.3 that

$$|z_L-z_R|\preccurlyeq|\zeta-\zeta_R|,$$

$$\left|\frac{\zeta-\zeta_R}{\zeta-z_2}\right|\asymp\frac{\delta}{t},$$

$$|\zeta-z_2|\asymp t^{2-\alpha}.$$

Finally we obtain

$$\frac{d}{t^{2-\alpha}} \asymp \frac{|z_L - z_R|}{t^{2-\alpha}} \leqslant \frac{|\zeta - \zeta_R|}{|\zeta - z_2|} \asymp \frac{\delta}{t}.$$

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